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Bijective linear maps on semimodules spanned by Boolean matrices of fixed rank

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ABSTRACT

Let $M_{m,n}(\mathcal{B})$ be the semimodule of all $m \times n$ Boolean matrices where $\mathcal{B}$ is the Boolean algebra with two elements. Let k be a positive integer such that $2 \leqslant k \leqslant \min(m, n)$. Let $\mathcal{B}(m, n, k)$ denote the subsemimodule of $M_{m,n}(\mathcal{B})$ spanned by the set of all rank k matrices. We show that if T is a bijective linear mapping on $\mathcal{B}(m, n, k)$, then there exist permutation matrices P and Q such that $T(A) = PAQ$ for all $A \in \mathcal{B}(m, n, k)$ or $m = n$ and $T(A) = PA^tQ$ for all $A \in \mathcal{B}(m, n, k)$. This result follows from a more general theorem we prove concerning the structure of linear mappings on $\mathcal{B}(m, n, k)$ that preserve both the weight of each matrix and rank one matrices of weight k^2. Here the weight of a Boolean matrix is the number of its non-zero entries.

1. Introduction

Let $\mathcal{B}$ denote the Boolean algebra with two elements 0 and 1. The addition and multiplication in $\mathcal{B}$ are defined as usual except that $1 + 1 = 1$. A matrix with entries from $\mathcal{B}$ is called a Boolean matrix. Let $M_{m,n}(\mathcal{B})$ be the semimodule of all $m \times n$ Boolean matrices. If A is an $m \times n$ non-zero Boolean matrix, its Boolean rank, $b(A)$, is the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of the zero matrix is 0. It is known that $b(A)$ is the least k such that A is the sum of k Boolean matrices of rank one (see [7]). A mapping T from a semimodule of Boolean matrices to another is called linear if T preserves sums and sends the zero matrix to the zero matrix.

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In [2], Beasley and Pullman proved the following result.

If T is a linear operator on $M_{m,n}(\mathcal{B})$, and $\min(m, n) \geqslant 2$, then the following statements are equivalent.

(i) T is invertible and preserves Boolean rank 1.
(ii) There exist permutation matrices P and Q such that $T(A) = PAQ$ for all $A \in M_{m,n}(\mathcal{B})$ or $m = n$ and $T(A) = PA^tQ$ for all $A \in M_{m,n}(\mathcal{B})$.

In [8], Pullman gave a graph-theoretic interpretation of the above result by using a result in [7] stating that the biclique covering number of a bipartite graph G and the Boolean rank of the (0,1)-incidence matrix of G are equal. In [5], Lim and Tan obtained several generalizations of the result of Beasley and Pullman.

Let k be a fixed integer such that $2 \leqslant k \leqslant \min(m, n)$. Let $\mathcal{B}(m, n, k)$ denote the subsemimodule of $M_{m,n}(\mathcal{B})$ spanned by the set of all rank k matrices. In this note, we first characterize linear mappings on $\mathcal{B}(m, n, k)$ that preserve both the weight of each matrix and rank one matrices of weight k^2. Here the weight of a Boolean matrix is the number of its non-zero entries. By using this characterization theorem, we show that if T is a bijective linear mapping on $\mathcal{B}(m, n, k)$, then there exist permutation matrices P and Q such that $T(A) = PAQ$ for all $A \in \mathcal{B}(m, n, k)$ or $m = n$ and $T(A) = PA^tQ$ for all $A \in \mathcal{B}(m, n, k)$. Using this characterization of bijective linear mappings on $\mathcal{B}(m, n, k)$, we describe the structure of bijective linear mappings on $M_{m,n}(\mathcal{B})$ that preserve rank k matrices of weight k. This result generalizes a theorem obtained by Song et al. [9] concerning the structure of bijective linear mappings on $M_{m,n}(\mathcal{B})$ that preserve rank k matrices.

2. Bijective linear mappings on semimodules spanned by Boolean matrices of fixed rank

Let I be a non-empty set. Let $\mathcal{B}_I$ denote the set of all functions f from I to $\mathcal{B}$ such that $\{i \in I : f(i) \neq 0\}$, the support of f, is a finite set. The cardinality of the support of f, denoted by $|f|$, is called the weight of f [3]. Every element of $\mathcal{B}_I$ with weight one is called a cell. For any $f, g \in \mathcal{B}_I$, let $f + g$ be the function from I to $\mathcal{B}$ such that $(f + g)(i) = f(i) + g(i)$ for any $i \in I$. Clearly $f + g \in \mathcal{B}_I$. For our purpose, we define a Boolean semimodule to be any subset of $\mathcal{B}_I$ containing the zero function which is closed under addition.

If f and g are in $\mathcal{B}_I$, we say that, f absorbs g, written $f \geqslant g$, if $f(i) \geqslant g(i)$ for any $i \in I$. Clearly $\mathcal{B}_I$ is a partially ordered set under this order relation.

Let U and V be Boolean semimodules. If $U \subseteq V$, then U is called a subsemimodule of V. Let S be a non-empty subset of U. Let $\langle S \rangle$ denote the intersection of all subsemimodules of U that contain S. Then $\langle S \rangle$ is a subsemimodule of U called the subsemimodule spanned by S. Note that $f \in \langle S \rangle$ if and only if f is a linear combination of a finite number of elements in S, i.e., $f = \sum_{i=1}^{k} \lambda_i s_i$ for some $s_1, \ldots, s_k$ in S and some λ_i in $\mathcal{B}$, $i = 1, \ldots, k$. The set S is called independent if $S \neq \{0\}$ when $|S| = 1$ or every element f in S is not a linear combination of any finite number of elements in $S \setminus \{f\}$ when $|S| \geqslant 2$. A subset E of U is called a basis of U if E is independent and $\langle E \rangle = U$. It is known that every Boolean semimodule has only one basis (see [3,5]). A mapping from U to V is called a (Boolean) linear mapping if it preserves sums and 0. If T is a linear mapping, then it preserves the order relation, i.e., $T(A) \geqslant T(B)$ whenever $A \geqslant B$.

For any $f \in \mathcal{B}_I$ and $g \in \mathcal{B}_J$, let $f \otimes g$ denote the function from $I \times J$ to $\mathcal{B}$ such that $(f \otimes g)(i, j) = f(i)g(j)$ for any $i \in I$ and $j \in J$. Then clearly $f \otimes g \in \mathcal{B}_{I\times J}$ and it is called a decomposable element. Clearly, $f \otimes g = 0$ if and only if $f = 0$ or $g = 0$.

Let s be a positive integer. Then a non-zero element A of $\mathcal{B}_{I\times J}$ is said to have rank s if A is the sum of s, but not less then s, non-zero decomposable elements.

For each $(i, j) \in I \times J$, let $E_{ij} \in \mathcal{B}_{I\times J}$ such that $E_{ij}(i, j) = 1$ and $|E_{ij}| = 1$.

A linear mapping $\theta : \mathcal{B}_I \to \mathcal{B}_J$ is called an embedding if θ sends distinct cells to distinct cells. Hence θ is an embedding if and only if there exists an injective mapping $\sigma : I \to J$ such that $\theta(e_i) = f_{\sigma(i)}$, $i \in I$ where $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$ are the bases of $\mathcal{B}_I$ and $\mathcal{B}_J$, respectively.

Let T be a linear mapping from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$. Then T is said to be induced by two linear mappings if one of the following conditions holds:

(i) There exist linear mappings $\theta : \mathcal{B}_I \to \mathcal{B}_M$ and $\varphi : \mathcal{B}_J \to \mathcal{B}_N$ such that $T(u \otimes v) = \theta(u) \otimes \varphi(v)$ for any u in $\mathcal{B}_I$ and any v in $\mathcal{B}_J$.
(ii) There exist linear mappings $\theta : \mathcal{B}_I \to \mathcal{B}_N$ and $\varphi : \mathcal{B}_J \to \mathcal{B}_M$ such that $T(u \otimes v) = \varphi(v) \otimes \theta(u)$ for any u in $\mathcal{B}_I$ and any v in $\mathcal{B}_J$.

If k is a positive integer and $k \leqslant \min(|I|, |J|)$, let $\mathcal{B}(I, J, k)$ denote the subsemimodule of $\mathcal{B}_{I\times J}$ spanned by all elements of rank k.

Lemma 2.1. *Let k be a positive integer $\geqslant 2$. Let $\mathcal{B}_1$ be the set of all rank k elements in $\mathcal{B}_{I\times J}$ with weight k and $\mathcal{B}_2$ be the set of all elements of the form*

$$E_{i_1j_1} + \cdots + E_{i_kj_k} + E_{i_1j_2} \tag{1}$$

where $i_1, \ldots, i_k$ are k distinct elements in I and $j_1, \ldots, j_k$ are k distinct elements in J. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ is the basis of $\mathcal{B}(I, J, k)$.

Proof. Let A be an element of the form (1). We shall show that A is of rank k. Let $B := E_{i_1j_1} + E_{i_1j_2}$, $C := E_{i_1j_2} + E_{i_2j_2}$. Since B is of rank one, it follows that A is the sum of k rank one elements and hence it is of rank $\leqslant k$. Suppose that A is of rank l and $0 < l < k$. Then $A = A_1 + \ldots + A_l$ for some rank one elements $A_1, \ldots, A_l$. Suppose that $k = 2$. Then $A = x \otimes y$ for some x in $\mathcal{B}_I$ and y in $\mathcal{B}_J$. Since $A \geqslant C$, we have $x(i_1) = x(i_2) = 1$. Since $A \geqslant B$, we have $y(j_1) = y(j_2) = 1$. Hence $|A| \geqslant 4$, a contradiction. Now suppose that $k > 2$. Since A does not absorb any rank one element of weight > 2, we have $|A_i| \leqslant 2$, $i = 1, \ldots, l$. Since $|A| = k + 1$ and $l < k$, there exist distinct integers s and t such that $|A_s| = |A_t| = 2$. Since A absorbs only two rank one elements B, C of weight 2, it follows that $\{A_s, A_t\} = \{B, C\}$ and $|A_i| = 1$ for $i \notin \{s, t\}$. This shows that $A_1 + \ldots + A_l$ has weight less than $k + 1$, a contradiction. Hence A is of rank k. Thus A belongs to $\mathcal{B}(I, J, k)$.

Let D be a rank k element of weight at least $k + 1$ in $\mathcal{B}_{I\times J}$. Then $D = u_1 \otimes v_1 + \ldots + u_k \otimes v_k$ for some non-zero elements u_i in $\mathcal{B}_I$ and some non-zero elements v_i in $\mathcal{B}_J$, $i = 1, \ldots, k$. Let U_i denote the set of all cells in $\mathcal{B}_I$ that are absorbed by u_i. Let H be a non-empty subset of $\{1, \ldots, k\}$. If $\bigcup_{i\in H} U_i$ consists of h cells $w_1, \ldots, w_h$ and $h < |H|$, then

$$\sum_{i\in H} u_i \otimes v_i = \sum_{i=1}^{h} w_i \otimes z_i$$

for some non-zero elements $z_1, \ldots, z_h$ in $\mathcal{B}_J$. This shows that D is the sum of $h + k - |H|$ rank one elements and hence the rank of D is less than k, a contradiction. Therefore $|\bigcup_{i\in H} U_i| \geqslant |H|$. By Hall's theorem, $U_1, \ldots, U_k$ has a system of distinct representatives $e_{s_1}, \ldots, e_{s_k}$. Similarly there exist k distinct cells $f_{t_1}, \ldots, f_{t_k}$ in $\mathcal{B}_J$ such that $f_{t_i} \leqslant v_i$, $\quad i = 1, \ldots, k$. Let $E := e_{s_1} \otimes f_{t_1} + \ldots + e_{s_k} \otimes f_{t_k}$. Then $D \geqslant E$ and $E \in \mathcal{B}_1$. Let $S := \{s_1, \ldots, s_k\}$ and $T := \{t_1, \ldots, t_k\}$. Let $(s, t) \in I \times J$ such that $D \geqslant E_{st}$ and E does not absorb E_{st}. If $(s, t) \in S \times T$, then $E + E_{st} \in \mathcal{B}_2$. If $(s, t) \notin S \times T$, then $E + E_{st}$ is the sum of two elements from $\mathcal{B}_1$. Hence in both cases, $E + E_{st} \in \langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$. Since D is the sum of elements of the form $E + E_{st}$, we see that $D \in \langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$. Hence $\mathcal{B}_1 \cup \mathcal{B}_2$ spans $\mathcal{B}(I, J, k)$. Since every element in $\mathcal{B}_2$ absorbs only one element from $\mathcal{B}_1$, it follows that no element in $\mathcal{B}_2$ is the sum of two elements from $\mathcal{B}_1$. Clearly no element from $\mathcal{B}_1$ can be the sum of some other elements from $\mathcal{B}_1 \cup \mathcal{B}_2$. Hence $\mathcal{B}_1 \cup \mathcal{B}_2$ is independent. This proves that $\mathcal{B}_1 \cup \mathcal{B}_2$ is the basis of $\mathcal{B}(I, J, k)$. $\square$

Theorem 2.2. *Let $T : \mathcal{B}(I, J, k) \to \mathcal{B}(M, N, k)$ be a linear mapping where* $\min(|I|, |J|, |M|, |N|) \geqslant k \geqslant 2$. *Then T satisfies the following two conditions*:

(i) $|T(A)| = |A|$ for any A in $\mathcal{B}(I, J, k)$,
(ii) T sends rank one matrices of weight k^2 to rank one matrices of weight k^2;

if and only if T can be extended to a linear mapping from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ which is induced by two embeddings.

Proof. The sufficiency is clear. We now prove the necessity. We first show that T is injective. Suppose that $T(A) = T(B)$. Then $T(A+B) = T(A) = T(B)$ and hence $|A+B| = |A| = |B|$ by hypothesis (i). This implies that $A \geqslant B$ and $B \geqslant A$, and thus $A = B$.

For convenience, let $U := \mathcal{B}(I,J,k)$ and $V := \mathcal{B}(M,N,k)$. We divide the remaining proof into the following two cases:

Case 1. $\{|I|, |J|\} \neq \{2\}$.
For each $(i,j) \in I \times J$, let

$$C_{ij} := \left\{A \in U : A \geqslant E_{ij}\right\},$$

and for each $(s,t) \in M \times N$, let

$$D_{st} := \{G \in V : G \geqslant E_{st}\}.$$

Claim 1. *For each $(i,j) \in I \times J$, $T(C_{ij}) \subseteq D_{st}$ for some $(s,t) \in M \times N$.*

Let $A \in C_{ij}$. There exist two elements B and C of weight k in U such that $B \geqslant E_{ij}$, $C \geqslant E_{ij}$ and $|B+C| = 2k-1$. Let $|A| = h$. Then

$$|A+B| = h+k-p \quad \text{and} \quad |A+C| = h+k-q$$

for some integers p and q, and hence $|A+B+C| = 2k+h-p-q$. Since $B \geqslant E_{ij}$, $C \geqslant E_{ij}$, and $|B+C| = 2k-1$, it follows that $T(B), T(C) \geqslant E_{st}$ for some $(s,t) \in M \times N$. Assume that $T(A) \not\geqslant E_{st}$. Since

$$|T(A)| = h, \quad |T(A)+T(B)| = h+k-p,$$
$$|T(A)+T(C)| = h+k-q$$

and

$$|T(B)+T(C)| = 2k-1, \quad |T(B)| = |T(C)| = k,$$

it follows that

$$|T(A)+T(B)+T(C)| = 2k+h-p-q-1 \neq |A+B+C|,$$

a contradiction to the hypothesis. This shows that $T(A) \geqslant E_{st}$ and hence $T(C_{ij}) \subseteq D_{st}$.

Claim 2. *If $(i,j) \in I \times J$, it is not possible that $T(C_{ij}) \subseteq D_{st}$ and $T(C_{ij}) \subseteq D_{mn}$ for some distinct (s,t), $(m,n) \in M \times N$.*

Suppose the contrary. Then $T(C_{ij}) \subseteq D_{st}$ and $T(C_{ij}) \subseteq D_{mn}$ for some distinct $(s,t), (m,n) \in M \times N$. Clearly there exist $B, C \in C_{ij}$ of weight k such that $|B+C| = 2k-1$. Since $|T(B)| = |T(C)| = k$, $T(B), T(C) \in D_{st}$ and $T(B), T(C) \in D_{mn}$, we have $|T(B)+T(C)| \leqslant 2k-2$, a contradiction to hypothesis (i). This proves Claim 2.

Claim 3. *If $T(C_{ij}) \subseteq D_{mn}$ and $T(C_{ih}) \subseteq D_{st}$, for $j \neq h$, then $m = s$ or $n = t$, but not both.*

There exists rank one element C of weight k^2 in U such that $C \geqslant E_{ij}$ and $C \geqslant E_{ih}$. In view of hypothesis (ii), $T(C)$ is a rank one element of weight k^2. We have

$$T(C) \geqslant E_{mn} \quad \text{and} \quad T(C) \geqslant E_{st}.$$

There exist $2d$ distinct elements $D_r \in U$, $r = 1, \ldots, 2d$, $d = (k-1)!$ each of weight k such that

$$C \geqslant D_r \quad \text{and} \quad D_r \geqslant E_{ij} \quad \text{or} \quad E_{ih}.$$

Suppose that $m \neq s$ and $n \neq t$. Then there are only $2d - (k-2)!$ distinct elements $E_r \in V$, $r = 1, \ldots, 2d - (k-2)!$, each of weight k, such that

$$T(C) \geqslant E_r \quad \text{and} \quad E_r \geqslant E_{mn} \quad \text{or} \quad E_{st},$$

a contradiction to the fact that T is injective. Hence $m = s$ or $n = t$.

Suppose that $m = s$ and $n = t$. Then there are only d distinct elements $F_r \in V$, $r = 1, \ldots, d$, each of weight k, such that

$$T(C) \geqslant F_r \quad \text{and} \quad F_r \geqslant E_{mn},$$

a contradiction to the fact that T is injective. This proves Claim 3.
Similarly, the following claim is true.

Claim 4. *If $T(C_{ij}) \subseteq D_{mn}$ and $T(C_{pj}) \subseteq D_{st}$, for $i \neq p$, then $m = s$ or $n = t$, but not both.*

Let L be the mapping from $I \times J$ to $M \times N$ be defined by

$$L(i,j) = (s,t) \quad \text{if } T(C_{ij}) \subseteq D_{st}.$$

In view of Claims 1–4, L is a well-defined mapping that sends any two ordered-pairs with exactly one equal component to two ordered-pairs with exactly one equal component. By a result of Westwick [10, Theorem 2.5], one of the following is true:

(i) $L(i,j) = (\sigma(i), \tau(j))$, $(i,j) \in I \times J$, for some injective mappings $\sigma : I \to M$, $\tau : J \to N$.
(ii) $L(i,j) = (\tau(j), \sigma(i))$, $(i,j) \in I \times J$, for some injective mappings $\sigma : I \to N$, $\tau : J \to M$.
(iii) $ImL \subseteq \{m\} \times N$ for some $m \in M$.
(iv) $ImL \subseteq M \times \{n\}$ for some $n \in N$.

(Westwick's result was proved under the condition that $\min(|I|, |J|) \geqslant 3$. However the result is also true when $\{|I|, |J|\} = \{2, c\}$ where $c \geqslant 3$). If (iii) or (iv) holds, then T maps rank k elements of weight k to rank one elements, a contradiction. Hence only (i) or (ii) holds. Suppose that (i) holds. Then for any non-element $X \in U$,

$$X = \sum_{(i,j)\in\Delta} E_{ij} \text{ for some non} - \text{empty subset } \Delta \subseteq I \times J.$$

Since X belongs to C_{ij} for every $(i,j) \in \Delta$, it follows that $T(X) \geqslant E_{\sigma(i)\tau(j)}$ for any $(i,j) \in \Delta$. Since $|T(X)| = |X|$, it follows that

$$T(X) = \sum_{(i,j)\in\Delta} E_{\sigma(i)\tau(j)}.$$

Let S be the linear mapping induced by two embeddings $\theta : \mathcal{B}_I \to \mathcal{B}_M$ and $\varphi : \mathcal{B}_J \to \mathcal{B}_N$ where

$$\theta(e_i) = g_{\sigma(i)}, \quad i \in I, \quad \varphi(f_j) = h_{\tau(j)}, \quad j \in J.$$

Here $\{e_i : i \in I\}$, $\{f_j : j \in J\}$, $\{g_m : m \in M\}$ and $\{h_n : n \in N\}$ are the bases of $\mathcal{B}_I, \mathcal{B}_J, \mathcal{B}_M$ and $\mathcal{B}_N$, respectively. It is now clear that the restriction of S to U is equal to T. Similarly, if (ii) holds, T can be extended to a linear mapping which is induced by two embeddings $\delta : \mathcal{B}_I \to \mathcal{B}_N$ and $\eta : \mathcal{B}_J \to \mathcal{B}_M$.

Case 2. $\{|I|, |J|\} = \{2\}$. We may assume that $I = J = \{1, 2\}$. Let $A_1 := E_{11} + E_{22}$, $A_2 := E_{12} + E_{21}$, $A_3 := A_1 + E_{12}$, $A_4 := A_1 + E_{21}$, $A_5 := A_2 + E_{11}$ and $A_6 := A_2 + E_{22}$. Then by Lemma 2.1, $\{A_1, \ldots, A_6\}$ is the basis of U. By hypothesis (i) and (ii),

$$T(A_1 + A_2) = E_{st} + E_{pq} + E_{sq} + E_{pt}$$

for some distinct $s, p \in M$ and some distinct $t, q \in N$. Let $B_1 := E_{st} + E_{pq}$, $B_2 := E_{sq} + E_{pt}$, $B_3 := B_1 + E_{sq}$, $B_4 := B_1 + E_{pt}$, $B_5 := B_2 + E_{st}$ and $B_6 := B_2 + E_{pq}$. Since T is an injective mapping that preserves the order relation and the weight of each element, we have

$$T(\{A_1, A_2\}) = \{B_1, B_2\} \quad \text{and} \quad T(\{A_3, \ldots, A_6\}) = \{B_3, \ldots, B_6\}.$$

Suppose that $T(A_1) = B_1$, then $T(A_2) = B_2$. We have

$$T(\{A_3, A_4\}) = \{B_3, B_4\} \quad \text{and} \quad T(\{A_5, A_6\}) = \{B_5, B_6\}$$

since $T(A_3), T(A_4) \geqslant B_1$ and $T(A_5), T(A_6) \geqslant B_2$. Similarly if $T(A_1) = B_2$, then $T(A_2) = B_1$, $T(\{A_3, A_4\}) = \{B_5, B_6\}$ and $T(\{A_5, A_6\}) = \{B_3, B_4\}$. This shows that there are exactly 8 linear mappings from U to V satisfying hypothesis (i) and (ii) with image equal to $W := \langle B_1, \ldots, B_6 \rangle$. Note that there are 8 embeddings $T_1, \ldots, T_8$ from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ with image $\langle E_{st}, E_{pq}, E_{sq}, E_{pt}\rangle$ which are induced by two embeddings. Since the restrictions of $T_1, \ldots, T_8$ to U are 8 linear mappings with image equal to W satisfying hypothesis (i) and (ii), it follows that T is the same as the restriction of one of $T_1, \ldots, T_8$ to U. $\square$

An $m \times s$ or $s \times m$ $(m \geqslant s)$ Boolean matrix is called a generalized permutation matrix if it is of rank s and weight s.

Corollary 2.3. *If T is a linear mapping from $\mathcal{B}(m, n, k)$ to $\mathcal{B}(p, q, k)$, $\min(m, n, p, q) \geqslant k \geqslant 2$, such that*

(i) $|T(A)| = |A|$ for any A in $\mathcal{B}(m, n, k)$
and
(ii) T sends rank one matrices of weight k^2 to rank one matrices of weight k^2,

then one of the following holds:

(a) There exist generalized permutation matrices P and Q of size $p \times m$ and size $n \times q$ respectively such that $T(A) = PAQ$ for all $A \in \mathcal{B}(m, n, k)$.
(b) There exist generalized permutation matrices P and Q of size $p \times n$ and size $m \times q$ respectively such that $T(A) = PA^tQ$ for all $A \in \mathcal{B}(m, n, k)$.

Theorem 2.4. *If $T : \mathcal{B}(I, J, k) \to \mathcal{B}(M, N, k)$ is a bijective linear mapping where $\min(|I|, |J|, |M|, |N|) \geqslant k \geqslant 2$, then $\{|I|, |J|\} = \{|M|, |N|\}$ and T can be extended to a linear mapping from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ which is induced by two bijective linear mappings.*

Proof. Let $A \in \mathcal{B}(I, J, k) \setminus \{0\}$. If $|A| = k$, then $|T(A)| = k$ since T is a bijective linear mapping. Suppose that $|A| = k + s$ with $s > 0$. Then $A > A_1$ for some $A_1 \in \mathcal{B}(I, J, k)$ of weight k. Clearly there exists a chain of $s + 1$ elements

$$A_1 < \cdots < A_{s+1}$$

from $\mathcal{B}(I, J, k)$ where $A_{s+1} = A$. Hence

$$T(A_1) < \cdots < T(A_{s+1})$$

and thus $|T(A)| \geqslant |A|$. Using T^{-1} we see that $|A| \geqslant |T(A)|$. Hence $|A| = |T(A)|$.

Let A be a rank one element of weight k^2 in $\mathcal{B}(I, J, k)$. Then there exist a set G consisting of k distinct elements $i_1, \ldots, i_k$ in I and a set H consisting of k distinct elements $j_1, \ldots, j_k$ in J such that $A = \sum_{(i,j)\in G\times H} E_{ij}$. Let $D := E_{i_1j_1} + \cdots + E_{i_kj_k}$. Let $m = k^2 - k$. Then there are m cells $C_1, \ldots, C_m$ in $\mathcal{B}_{I\times J}$ such that $A = D + C_1 + \ldots + C_m$. For each $i = 1, \ldots, m$, let $D_i = D + C_i$. Then $A = \sum_{i=1}^{m} D_i$ and $D_i \geqslant D$. By Lemma 2.1, D is an element of weight k in the basis of $\mathcal{B}(I, J, k)$. Since T is a bijective linear mapping, it follows that $T(D)$ is an element of weight k in the basis of $\mathcal{B}(M, N, k)$. Hence

$$T(D) = E_{p_1q_1} + \cdots + E_{p_kq_k}$$

for some k distinct elements $p_1, \ldots, p_k$ in M and some k distinct elements $q_1, \ldots, q_k$ in N. Since D_i is an element of weight $k + 1$ in the basis of $\mathcal{B}(I, J, k)$ and T is bijective, it follows that $T(D_i)$ is an element of weight $k + 1$ in the basis of $\mathcal{B}(M, N, k)$. For each $i = 1, \ldots, m$, we have $T(D_i) \geqslant T(D)$. Hence

$$T(D_i) = T(D) + E_{s_it_i}$$

for some $s_i \in \{p_1, \ldots, p_k\}$, $t_i \in \{q_1, \ldots, q_k\}$ where $(s_i, t_i) \neq (p_j, q_j)$ for $j = 1, \ldots, k$. Since T is injective, we have $(s_i, t_i) \neq (s_h, t_h)$ for any distinct i and h. Since $T(A) = \sum_{i=1}^{m} T(D_i)$, it follows that $T(A)$ is the sum of all E_{st} where $s \in \{p_1, \ldots, p_k\}$ and $t \in \{q_1, \ldots, q_k\}$. This shows that $T(A) = x \otimes y$ where $x(p_i) = y(q_i) = 1, i = 1, \ldots, k$, and $|x| = |y| = k$. Hence $T(A)$ is of rank one. The result now follows from Theorem 2.2. $\square$

Corollary 2.5. *If T is a bijective linear mapping from $\mathcal{B}(m,n,k)$ to $\mathcal{B}(p,q,k)$ where k is a fixed positive integer such that* $\min(m,n,p,q) \geqslant k \geqslant 2$, *then $\{m,n\} = \{p,q\}$ and there exist $m \times m$ permutation matrix P and $n \times n$ permutation matrix Q such that $T(A) = PAQ$ for all A in $\mathcal{B}(m,n,k)$ or $T(A) = QA^tP$ for all A in $\mathcal{B}(m,n,k)$.*

We now use Theorem 2.4 to characterize bijective linear mappings that preserve rank k elements of weight k.

Theorem 2.6. *If T is a bijective linear mapping from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ that sends rank k elements of weight k to rank k elements of weight k where k is a fixed integer $\geqslant 2$ and* $\min(|I|,|J|,|M|,|N|) \geqslant k$, *then $\{|I|,|J|\} = \{|M|,|N|\}$ and T is induced by two bijective linear mappings.*

Proof. Let A be a rank k element in $\mathcal{B}_{I\times J}$. If A has weight k, then by hypothesis, $T(A)$ belongs to $\mathcal{B}(M,N,k)$. If A has weight $k+1$, then $T(A)$ is the sum of a rank k element with weight k and a cell in $\mathcal{B}_{M\times N}$ and hence it belongs $\mathcal{B}(M,N,k)$. Since T is linear and $\mathcal{B}(M,N,k)$ is spanned by rank k elements of weight either k or $k+1$, it follows that

$$T(\mathcal{B}(I,J,k)) \subseteq \mathcal{B}(M,N,k).$$

Let D be a rank k element of weight k in $\mathcal{B}_{M\times N}$. Since T bijectively maps the basis of $\mathcal{B}_{I\times J}$ onto the basis of $\mathcal{B}_{M\times N}$, there is an element C of weight k such that $T(C) = D$. Suppose that C is not of rank k. Then there exist distinct cells C_1, C_2 of $\mathcal{B}_{I\times J}$ such that $C \geqslant C_1, C_2$ and $C_1 + C_2$ is of rank one. Without loss of generality we may assume that $C_1 = E_{ck}$, $C_2 = E_{cl}$ for some $c \in I$ and some distinct $k, l \in J$. Let $T(C_1) = E_{st}$, $T(C_2) = E_{mn}$. Then $s \neq m$, $t \neq n$. Let U be the subsemimodule of $\mathcal{B}_{I\times J}$ spanned by all cells E_{ij}, $i \neq c$, $j \neq k$ and V be the subsemimodule of $\mathcal{B}_{M\times N}$ spanned by all cells E_{pq}, $p \neq s$, $q \neq t$. For any X in U of rank $k-1$ and weight $k-1$, $T(C_1+X)$ is of rank k and weight k and hence $T(X) \in V$. This shows that $T(U) \subseteq V$. Let $d \in I\setminus\{c\}$. Since $E_{dl} \in U$, it follows that $T(E_{dl}) = E_{ry}$ for some $r \neq s$, $y \neq t$. Since$T$ sends the basis of $\mathcal{B}_{I\times J}$ onto the basis of $\mathcal{B}_{M\times N}$, it follows that

$$T(A) = E_{sy} \quad \text{and} \quad T(B) = E_{rt}$$

for some distinct cells A, B in $\mathcal{B}_{I\times J}$. Either $A \neq E_{dk}$ or $B \neq E_{dk}$. We may assume that $A \neq E_{dk}$. Since $A \notin U$, we obtain that $A + E_{dl}$ is of rank 2. Choose a rank $k-2$ element R of weight $k-2$ in $\mathcal{B}_{I\times J}$ such that $A + E_{dl} + R$ is of rank k. We see that $T(A + E_{dl} + R)$ is of rank less than k, a contradiction to the hypothesis. Hence C is a rank k element with weight k such that $T(C) = D$.

Now let G be a rank k element of weight $k+1$ in $\mathcal{B}_{M\times N}$. Then $G = P + Q$ for some rank k element P of weight k and some cell Q in $\mathcal{B}_{M\times N}$. By our previous argument, there exists a rank k element A of weight k such that $T(A) = P$. Let B be the cell such that $T(B) = Q$. Then $T(A+B) = G$. Note that $A + B \in \mathcal{B}(I,J,k)$.

Since $\mathcal{B}(M,N,k)$ is spanned by rank k elements of weight either k or $k+1$, we see that $T(\mathcal{B}(I,J,k)) = \mathcal{B}(M,N,k)$. By Theorem 2.4, there is a linear mapping S from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ which is induced by two bijective linear mappings such that $T(A) = S(A)$ for any $A \in \mathcal{B}(I,J,k)$. Now for any cell E in $\mathcal{B}_{I\times J}$, there exists an element F in $\mathcal{B}_{I\times J}$ of rank k and weight k such that $E + F$ is of rank k and weight $k+1$. Since

$$T(E+F) = S(E+F), \quad T(F) = S(F),$$

it follows that $T(E) = S(E)$. This shows that $T = S$ and the proof is complete. $\square$

Corollary 2.7. *If T is a bijective linear mapping from $M_{m,n}(\mathcal{B})$ to $M_{p,q}(\mathcal{B})$ that sends rank k matrices of weight k to rank k matrices of weight k where k is a fixed positive integer such that* $\min(m,n,p,q) \geqslant k \geqslant 2$, *then $\{m,n\} = \{p,q\}$ and there exist $m \times m$ permutation matrix P and $n \times n$ permutation matrix Q such that $T(A) = PAQ$ for all $A \in M_{m,n}(\mathcal{B})$ or $T(A) = QA^tP$ for all $A \in M_{m,n}(\mathcal{B})$.*

The following result was proved in [9] when both $\mathcal{B}_{I\times J}$ and $\mathcal{B}_{M\times N}$ are the same semimodule of all $m \times n$ Boolean matrices. It follows immediately from Theorem 2.6 since every bijective linear rank k preserver sends rank k elements of weight k to rank k elements of weight k.

Corollary 2.8. *If T is a bijective linear mapping from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ that sends rank k elements to rank k elements where k is a fixed integer $\geqslant 2$ and $\min(|I|, |J|, |M|, |N|) \geqslant k$, then $\{|I|, |J|\} = \{|M|, |N|\}$ and T is induced by two bijective linear mappings.*

Corollary 2.9. *If L is a bijective linear mapping from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$. Let k be a fixed integer such that $\min(|I|, |J|, |M|, |N|) \geqslant k \geqslant 2$. If L sends elements of rank $< k$ to elements of rank $< k$, then $\{|I|, |J|\} = \{|M|, |N|\}$ and L is induced by two bijective linear mappings.*

Proof. Let $T := L^{-1}$. Let A be an element of rank k and weight k in $\mathcal{B}_{M\times N}$. Since T sends the basis of $\mathcal{B}_{M\times N}$ onto the basis of $\mathcal{B}_{I\times J}$, it follows that $T(A)$ is the sum of k cells in $\mathcal{B}_{I\times J}$. By hypothesis, the rank of $T(A)$ cannot be less than k.Hence $T(A)$ is of rank k and weight k. The result now follows from Theorem 2.6. $\square$

Remark 2.10. An embedding from $\mathcal{B}_{I\times J}$ to $\mathcal{B}_{M\times N}$ that sends rank k elements of weight k to rank k elements of weight k, where k is a fixed integer $\geqslant 2$, is not necessarily induced by two embeddings. For example, let $T : M_2(\mathcal{B}) \to M_3(\mathcal{B})$ be the linear mapping defined as follows:

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 & b \\ 0 & d & 0 \\ 0 & c & 0 \end{bmatrix}.$$

Then T is an embedding that sends rank 2 matrices of weight 2 to rank 2 matrices of weight 2. However, there do not exist matrices P and Q such that $T(A) = PAQ$ for all $A \in M_2(\mathcal{B})$ or $T(A) = PA^tQ$ for all $A \in M_2(\mathcal{B})$ since T sends the rank one matrix $E_{12} + E_{22}$ to a rank 2 matrix. Note that T also sends rank 2 matrices to rank 2 matrices.

Example 2.11. Let N denote the set of all positive integers. Let T be the linear mapping on $\mathcal{B}_{N\times N}$ such that $T(E_{ij}) = E_{ii}$ for all $i, j \in N$. Then T preserves rank k elements of weight k for any positive integer k. However, T is not induced by any two linear mappings since T sends the rank one element $E_{11} + E_{12} + E_{21} + E_{22}$ to the rank 2 element $E_{11} + E_{22}$. Note that T is not a rank k preserver since T sends the rank k element $E_{11} + E_{12} + E_{21} + E_{22} + E_{33} + \ldots + E_{k+1,k+1}$ to the rank $k+1$ element $E_{11} + \ldots. + E_{k+1,k+1}$.

There are many types of linear rank k preservers from one semimodule of Boolean matrices to another which are not of the standard induced forms. We mention two examples as follows:

Example 2.12. Let U, V, R and S be four n-square permutation matrices where $n \geqslant 2$. Let T be the mapping on $M_{2n}(\mathcal{B})$ defined as follows:

$$T\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\right) = \left[\begin{array}{c|c} UAV & 0 \\ \hline 0 & RBS \end{array}\right],$$

$$T\left(\left[\begin{array}{c|c} A & C \\ \hline D & B \end{array}\right]\right) = \left[\begin{array}{c|c} J & 0 \\ \hline 0 & J \end{array}\right]$$

if $C \neq 0$ or $D \neq 0$, where $A, B \in M_n(\mathcal{B})$ and J is the n-square matrix with all entries equal to 1. Then T is a linear rank 2 preserver. However, there do not exist matrices P and Q such that $T(A) = PAQ$ for all $A \in M_{2n}(\mathcal{B})$ or $T(A) = QA^tP$ for all $A \in M_{2n}(\mathcal{B})$ since $T(E_{11} + E_{1,n+1})$ is of rank 2. Note that one could easily extend this construction to linear rank k preservers on $M_{kn}(\mathcal{B})$ for any positive integer k.

Example 2.13. Let m, n, p, q, k be integers such that $m, n > k \geqslant 2, p \geqslant k, q > k + 2$. Let $A_i := E_{1i} + E_{2,i+1} + \ldots + E_{k,i+k-1}, i = 1, \ldots, 4$, be 4 matrices in $M_{p,q}(\mathcal{B})$. Then every non-zero matrix in $\langle A_1, \ldots, A_4\rangle$ is of rank k. Let T be the linear mapping from $M_{m,n}(\mathcal{B})$ to $M_{p,q}(\mathcal{B})$ such that

$$T(E_{st}) = A_1 \quad \text{if } s, t \leqslant k,\ T(E_{st}) = A_2 \quad \text{if } s \leqslant k, t > k,$$
$$T(E_{st}) = A_3 \quad \text{if } s > k,\ t \leqslant k, \quad T(E_{st}) = A_4 \quad \text{if } s, t > k.$$

Then T sends every non-zero matrix to a rank k matrix and hence it is a rank k preserver. Clearly T is not of the standard induced form.

Remark 2.14. The problem concerning characterization of linear rank k preservers on various spaces of matrices over a field has been studied by many authors. Marcus and Moyls [6] proved the following result: Every linear rank one preserver T on the space of all $m \times n$ complex matrices takes one of the following forms: (i) $T(A) = PAQ$ for all A; (ii) $m = n$ and $T(A) = PA^tQ$ for all A where P and Q are invertible matrices. They conjectured that the above result is true for any linear rank k preserver where $k \leqslant \min(m, n)$. This conjecture was studied by a number of authors and was eventually resolved by Beasley [1]. For a survey of this area of work, see [4].

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